Simple trigonometric substitutions with broad results

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Often, the key to solve some intricate algebraic inequality is to simplify it by employing a trigonometric substitution. When we make a clever trigonometric substitution the problem may reduce so much that we can see a direct solution immediately. Besides, trigonometric functions have well-known properties that may help in solving such inequalities. As a result, many algebraic problems can be solved by using an inspired substitution.

We start by introducing the readers to such substitutions. After that we present some well-known trigonometric identities and inequalities. Finally, we discuss some Olympiad problems and leave others for the reader to solve.

**Theorem 1.** Let \( \alpha, \beta, \gamma \) be angles in \((0, \pi)\). Then \( \alpha, \beta, \gamma \) are the angles of a triangle if and only if

\[
\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} = 1.
\]

**Proof.** First of all note that if \( \alpha = \beta = \gamma \), then the statement clearly holds. Assume without loss of generality that \( \alpha \neq \beta \). Because \( 0 < \alpha + \beta < 2\pi \), it follows that there exists an angle in \((-\pi, \pi)\), say \( \gamma' \), such that \( \alpha + \beta + \gamma' = \pi \).

Using the addition formulas and the fact that \( \tan x = \cot \left( \frac{\pi}{2} - x \right) \), we have

\[
\tan \frac{\gamma'}{2} = \cot \frac{\alpha + \beta}{2} = \frac{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}},
\]

yielding

\[
\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma'}{2} + \tan \frac{\gamma'}{2} \tan \frac{\alpha}{2} = 1. \tag{1}
\]

Now suppose that

\[
\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} = 1, \tag{2}
\]

for some \( \alpha, \beta, \gamma \) in \((0, \pi)\).

We will prove that \( \gamma = \gamma' \), and this will imply that \( \alpha, \beta, \gamma \) are the angles of a triangle. Subtracting (1) from (2) we get \( \tan \frac{\gamma}{2} = \tan \frac{\gamma'}{2} \). Thus \( \left| \frac{\gamma - \gamma'}{2} \right| = k\pi \) for some nonnegative integer \( k \). But \( \left| \frac{\gamma - \gamma'}{2} \right| \leq \left| \frac{\gamma}{2} \right| + \left| \frac{\gamma'}{2} \right| < \pi \), so it follows that \( k = 0 \).

That is \( \gamma = \gamma' \), as desired. \( \square \)
Theorem 2. Let $\alpha, \beta, \gamma$ be angles in $(0, \pi)$. Then $\alpha, \beta, \gamma$ are the angles of a triangle if and only if

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = 1.$$ 

Proof. As $0 < \alpha + \beta < 2\pi$, there exists an angle in $(-\pi, \pi)$, say $\gamma'$, such that $\alpha + \beta + \gamma' = \pi$. Using the product-to-sum and the double angle formulas we get

$$\sin^2 \frac{\gamma'}{2} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma'}{2} = \cos \frac{\alpha + \beta}{2} \left( \cos \frac{\alpha + \beta}{2} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right)$$

$$= \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$= \cos \alpha + \cos \beta$$

$$= \frac{(1 - 2 \sin^2 \frac{\alpha}{2}) + (1 - 2 \sin^2 \frac{\beta}{2})}{2}$$

$$= 1 - \sin^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2}.$$ 

Thus

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma'}{2} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma'}{2} = 1.$$ 

(1)

Now suppose that

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = 1,$$ 

(2)

for some $\alpha, \beta, \gamma$ in $(0, \pi)$. Subtracting (1) from (2) we obtain

$$\sin^2 \frac{\gamma}{2} - \sin^2 \frac{\gamma'}{2} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \left( \sin \frac{\gamma}{2} - \sin \frac{\gamma'}{2} \right) = 0,$$

that is

$$\left( \sin \frac{\gamma}{2} - \sin \frac{\gamma'}{2} \right) \left( \sin \frac{\gamma}{2} + \sin \frac{\gamma'}{2} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) = 0.$$ 

The second factor can be written as

$$\sin \frac{\gamma}{2} + \sin \frac{\gamma'}{2} + \cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2} = \sin \frac{\gamma}{2} + \cos \frac{\alpha - \beta}{2},$$

which is clearly greater than 0. It follows that $\sin \frac{\gamma}{2} = \sin \frac{\gamma'}{2}$, and so $\gamma = \gamma'$, showing that $\alpha, \beta, \gamma$ are the angles of a triangle. □
Substitutions and Transformations

**T1.** Let $\alpha, \beta, \gamma$ be angles of a triangle. Let

$$A = \frac{\pi - \alpha}{2}, \quad B = \frac{\pi - \beta}{2}, \quad C = \frac{\pi - \gamma}{2}. \quad \text{Then } A + B + C = \pi, \text{ and } 0 \leq A, B, C < \frac{\pi}{2}.$$

This transformation allows us to switch from angles of an arbitrary triangle to angles of an acute triangle. Note that

- $\cyc(\sin \frac{\alpha}{2} = \cos A)$,  
- $\cyc(\cos \frac{\alpha}{2} = \sin A)$,  
- $\cyc(\tan \frac{\alpha}{2} = \cot A)$,  
- $\cyc(\cot \frac{\alpha}{2} = \tan A)$,

where by $\cyc$ we denote a cyclic permutation of angles.

**T2.** Let $x, y, z$ be positive real numbers. Then there is a triangle with sidelengths $a = x + y$, $b = y + z$, $c = z + x$. This transformation is sometimes called Dual Principle. Clearly, $s = x + y + z$ and $(x, y, z) = (s-a, s-b, s-c)$. This transformation already triangle inequality.

**S1.** Let $a, b, c$ be positive real numbers such that $ab + bc + ca = 1$. Using the function $f : (0, \frac{\pi}{2}) \to (0, +\infty)$, for $f(x) = \tan x$, we can do the following substitution

$$a = \tan \frac{\alpha}{2}, \quad b = \tan \frac{\beta}{2}, \quad c = \tan \frac{\gamma}{2},$$

where $\alpha, \beta, \gamma$ are the angles of a triangle $ABC$.

**S2.** Let $a, b, c$ be positive real numbers such that $ab + bc + ca = 1$. Applying **T1** to **S1**, we have

$$a = \cot A, \quad b = \cot B, \quad c = \cot C,$$

where $A, B, C$ are the angles of an acute triangle.

**S3.** Let $a, b, c$ be positive real numbers such that $a + b + c = abc$. Dividing by $abc$ it follows that $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = 1$. Due to **S1**, we can substitute

$$\frac{1}{a} = \tan \frac{\alpha}{2}, \quad \frac{1}{b} = \tan \frac{\beta}{2}, \quad \frac{1}{c} = \tan \frac{\gamma}{2},$$

that is

$$a = \cot \frac{\alpha}{2}, \quad b = \cot \frac{\beta}{2}, \quad c = \cot \frac{\gamma}{2},$$

where $\alpha, \beta, \gamma$ are the angles of a triangle.

**S4.** Let $a, b, c$ be positive real numbers such that $a + b + c = abc$. Applying **T1** to **S3**, we have

$$a = \tan A, \quad b = \tan B, \quad c = \tan C,$$
where $A, B, C$ are the angles of an acute triangle.

**S5.** Let $a, b, c$ be positive real numbers such that $a^2 + b^2 + c^2 + 2abc = 1$. Note that since all the numbers are positive it follows that $a, b, c < 1$. Using the function $f : (0, \pi) \to (0, 1)$, for $f(x) = \sin \frac{x}{2}$, and recalling Theorem 2, we can substitute

$$a = \sin \frac{\alpha}{2}, \quad b = \sin \frac{\beta}{2}, \quad c = \sin \frac{\gamma}{2},$$

where $\alpha, \beta, \gamma$ are the angles of a triangle.

**S6.** Let $a, b, c$ be positive real numbers such that $a^2 + b^2 + c^2 + 2abc = 1$. Applying $T_1$ to $S5$, we have

$$a = \cos A, \quad b = \cos B, \quad c = \cos C,$$

where $A, B, C$ are the angles of an acute triangle.

**S7.** Let $x, y, z$ be positive real numbers. Applying $T_2$ to expressions

$$\sqrt{\frac{yz}{(x+y)(x+z)}}, \quad \sqrt{\frac{xz}{(y+z)(y+x)}}, \quad \sqrt{\frac{xy}{(z+x)(z+y)}},$$

they can be substituted by

$$\sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \sqrt{\frac{(s-c)(s-a)}{ca}}, \quad \sqrt{\frac{(s-a)(s-b)}{ab}},$$

where $a, b, c$ are the sidelengths of a triangle. Recall the following identities

$$\sin \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

Thus our expressions can be substituted by

$$\sin \frac{\alpha}{2}, \quad \sin \frac{\beta}{2}, \quad \sin \frac{\gamma}{2},$$

where $\alpha, \beta, \gamma$ are the angles of a triangle.

**S8.** Analogously to $S7$, the expressions

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}, \quad \sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}}, \quad \sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}},$$

can be substituted by

$$\cos \frac{\alpha}{2}, \quad \cos \frac{\beta}{2}, \quad \cos \frac{\gamma}{2},$$

where $\alpha, \beta, \gamma$ are the angles of a triangle.
Further we present a list of inequalities and equalities that can be helpful in solving many problems or simplify them.

Well-known inequalities

Let \( \alpha, \beta, \gamma \) be angles of a triangle \( ABC \). Then

1. \( \cos \alpha + \cos \beta + \cos \gamma \leq \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2} \)

2. \( \sin \alpha + \sin \beta + \sin \gamma \leq \cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \leq \frac{3\sqrt{3}}{2} \)

3. \( \cos \alpha \cos \beta \cos \gamma \leq \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \leq \frac{1}{8} \)

4. \( \sin \alpha \sin \beta \sin \gamma \leq \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \leq \frac{3\sqrt{3}}{8} \)

5. \( \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{C}{2} \geq 3\sqrt{3} \)

6. \( \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma \geq \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{C}{2} \geq \frac{3}{4} \)

7. \( \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \leq \cos^2 \frac{\alpha}{2} + \cos^2 \frac{\beta}{2} + \cos^2 \frac{\gamma}{2} \leq \frac{9}{4} \)

8. \( \cot \alpha + \cot \beta + \cot \gamma \geq \tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \geq \sqrt{3} \)

Well-known identities

Let \( \alpha, \beta, \gamma \) be angles of a triangle \( ABC \). Then

1. \( \cos \alpha + \cos \beta + \cos \gamma = 1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \)

2. \( \sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \)

3. \( \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma \)

4. \( \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2 + 2 \cos \alpha \cos \beta \cos \gamma \)

For arbitrary angles \( \alpha, \beta, \gamma \) we have

\[
\sin \alpha + \sin \beta + \sin \gamma - \sin(\alpha + \beta + \gamma) = 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2}.
\]

\[
\cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha + \beta + \gamma) = 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2}.
\]
Applications

1. Let \( x, y, z \) be positive real numbers. Prove that
\[
\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(y+z)(y+x)}} + \frac{z}{z + \sqrt{(z+x)(z+y)}} \leq 1.
\]
(Walther Janous, Crux Mathematicorum)

Solution. The inequality is equivalent to
\[
\sum \frac{1}{1 + \sqrt{(x+y)(x+z)}} \leq 1.
\]
Because the inequality is homogeneous, we can assume that \( xy + yz + zx = 1 \).
Let us apply substitution \( S_1: \text{cyc}(x = \tan \frac{\alpha}{2}) \), where \( \alpha, \beta, \gamma \) are angles of a triangle. We get
\[
\frac{(x+y)(x+z)}{x^2} = \left( \tan \frac{\alpha}{2} + \tan \frac{\beta}{2} \right) \left( \tan \frac{\alpha}{2} + \tan \frac{\gamma}{2} \right) = \frac{1}{\sin^2 \frac{\alpha}{2}},
\]
and similar expressions for the other terms. The inequality becomes
\[
\frac{\sin \frac{\alpha}{2}}{1 + \sin \frac{\alpha}{2}} + \frac{\sin \frac{\beta}{2}}{1 + \sin \frac{\beta}{2}} + \frac{\sin \frac{\gamma}{2}}{1 + \sin \frac{\gamma}{2}} \leq 1,
\]
that is
\[
2 \leq \frac{1}{1 + \sin \frac{\alpha}{2}} + \frac{1}{1 + \sin \frac{\beta}{2}} + \frac{1}{1 + \sin \frac{\gamma}{2}}.
\]
On the other hand, using the well-known inequality \( \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2} \)
and the Cauchy-Schwarz inequality, we have
\[
2 \leq \frac{9}{(1 + \sin \frac{\alpha}{2}) + (1 + \sin \frac{\beta}{2}) + (1 + \sin \frac{\gamma}{2})} \leq \sum \frac{1}{1 + \sin \frac{\alpha}{2}},
\]
and we are done. \( \blacksquare \)

2. Let \( x, y, z \) be real numbers greater than 1 such that \( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2 \). Prove that
\[
\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \leq \sqrt{x+y+z}.
\]
(Iran, 1997)
Solution. Let \((x, y, z) = (a + 1, b + 1, c + 1)\), with \(a, b, c\) positive real numbers. Note that the hypothesis is equivalent to \(ab + bc + ca + 2abc = 1\). Then it suffices to prove that
\[
\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \sqrt{a + b + c + 3}.
\]
Squaring both sides of the inequality and canceling some terms yields
\[
\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \leq \frac{3\sqrt{2}}{2}.
\]
Using substitution \(S5\) we get \(ab, bc, ca = (\sin^2 \frac{\alpha}{2}, \sin^2 \frac{\beta}{2}, \sin^2 \frac{\gamma}{2})\), where \(ABC\) is an arbitrary triangle. The problem reduces to proving that
\[
\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} \leq \frac{3\sqrt{2}}{2},
\]
which is well-known and can be done using Jensen inequality.

3. Let \(a, b, c\) be positive real numbers such that \(a + b + c = 1\). Prove that
\[
\sqrt{\frac{ab}{c + ab}} + \sqrt{\frac{bc}{a + bc}} + \sqrt{\frac{ca}{b + ca}} \leq \frac{3\sqrt{2}}{2}.
\]
\((Open Olympiad of FML No-239, Russia)\)

Solution. The inequality is equivalent to
\[
\sqrt{\frac{ab}{(c + a)(c + b)}} + \sqrt{\frac{bc}{(a + b)(a + c)}} + \sqrt{\frac{ca}{(b + c)(b + a)}} \leq \frac{3\sqrt{2}}{2}.
\]
Substitution \(S7\) replaces the three terms in the inequality by \(\sin^2 \frac{\alpha}{2}, \sin^2 \frac{\beta}{2}, \sin^2 \frac{\gamma}{2}\). Thus it suffices to prove \(\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} \leq \frac{3\sqrt{2}}{2}\), which clearly holds.

4. Let \(a, b, c\) be positive real numbers such that \((a + b)(b + c)(c + a) = 1\). Prove that
\[
ab + bc + ca \leq \frac{3}{4}.
\]
\((Cezar Lupu, Romania, 2005)\)

Solution. Observe that the inequality is equivalent to
\[
\left(\sum ab\right)^3 \leq \left(\frac{3}{4}\right)^3 (a + b)^2(b + c)^2(c + a)^2.
\]
Because the inequality is homogeneous, we can assume that \( ab + bc + ca = 1 \). We use substitution \( S1: \cyc(a = \tan \frac{\alpha}{2}) \), where \( \alpha, \beta, \gamma \) are the angles of a triangle. Note that

\[
(a + b)(b + c)(c + a) = \prod \left( \frac{\cos \frac{\gamma}{2}}{\cos \frac{\alpha}{2} \cos \frac{\beta}{2}} \right) = \frac{1}{\cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}}.
\]

Thus it suffices to prove that

\[
\left( \frac{4}{3} \right)^3 \leq \frac{1}{\cos^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2}},
\]
or

\[
4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \leq \frac{3\sqrt{3}}{2}.
\]

From the identity \( 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} = \sin \alpha + \sin \beta + \sin \gamma \), the inequality is equivalent to

\[
\sin \alpha + \sin \beta + \sin \gamma \leq \frac{3\sqrt{3}}{2}.
\]

But \( f(x) = \sin x \) is a concave function on \((0, \pi)\) and the conclusion follows from Jensen’s inequality. ■

5. Let \( a, b, c \) be positive real numbers such that \( a + b + c = 1 \). Prove that

\[
a^2 + b^2 + c^2 + 2\sqrt{3}abc \leq 1.
\]

(Poland, 1999)

**Solution.** Let \( \cyc \left( x = \sqrt{\frac{bc}{a}} \right) \). It follows that \( \cyc(a = yz) \). The inequality becomes

\[
x^2y^2 + y^2z^2 + x^2z^2 + 2\sqrt{3}xyz \leq 1,
\]

where \( x, y, z \) are positive real numbers such that \( xy + yz + zx = 1 \). Note that the inequality is equivalent to

\[
(xy + yz + zx)^2 + 2\sqrt{3}xyz \leq 1 + 2xyz(x + y + z),
\]
or

\[
\sqrt{3} \leq x + y + z.
\]

Applying substitution \( S1 \cyc(x = \tan \frac{\alpha}{2}) \), it suffices to prove

\[
\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \geq \sqrt{3}.
\]

The last inequality clearly holds, as \( f(x) = \tan \frac{x}{2} \) is convex function on \((0, \pi)\), and the conclusion follows from Jensen’s inequality. ■
6. Let \( x, y, z \) be positive real numbers. Prove that

\[
\sqrt{x(y + z)} + \sqrt{y(z + x)} + \sqrt{z(x + y)} \geq 2 \sqrt{\frac{(x + y)(y + z)(z + x)}{x + y + z}}
\]

\( \text{(Darij Grinberg)} \)

**Solution.** Rewrite the inequality as

\[
\sqrt{\frac{x(x + y + z)}{(x + y)(x + z)}} + \sqrt{\frac{y(x + y + z)}{(y + z)(y + x)}} + \sqrt{\frac{z(x + y + z)}{(z + x)(z + y)}} \geq 2.
\]

Applying substitution \( S8 \), it suffices to prove that

\[
\cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \geq 2,
\]

where \( \alpha, \beta, \gamma \) are the angles of a triangle. Using transformation \( T1 \) \( \text{cyc}(A = \frac{\pi - \alpha}{2}) \), where \( A, B, C \) are angles of an acute triangle, the inequality is equivalent to

\[
\sin A + \sin B + \sin C \geq 2.
\]

There are many ways to prove this fact. We prefer to use Jordan’s inequality, that is

\[
\frac{2\alpha}{\pi} \leq \sin \alpha \leq \alpha \quad \text{for all } \alpha \in (0, \frac{\pi}{2}).
\]

The conclusion immediately follows. \( \blacksquare \)

7. Let \( a, b, c \) be positive real numbers such that \( a + b + c + 1 = 4abc \). Prove that

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3 \geq \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}.
\]

\( \text{(Daniel Campos Salas, Mathematical Reflections, 2007)} \)

**Solution.** Rewrite the condition as

\[
\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} + \frac{1}{abc} = 4.
\]

Observe that we can use substitution \( S5 \) in the following way

\[
\left( \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab} \right) = \left( 2 \sin^2 \frac{\alpha}{2}, 2 \sin^2 \frac{\beta}{2}, 2 \sin^2 \frac{\gamma}{2} \right),
\]
where \( \alpha, \beta, \gamma \) are angles of a triangle. It follows that
\[
\left( \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right) = \left( \frac{2 \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sin \frac{\alpha}{2}}, \frac{2 \sin \frac{\gamma}{2} \sin \frac{\alpha}{2}}{\sin \frac{\beta}{2}}, \frac{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sin \frac{\gamma}{2}} \right).
\]

Then it suffices to prove that
\[
\sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} + \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sin \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \geq \frac{3}{2} \geq \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2}.
\]

The right-hand side of the inequality is well known. For the left-hand side we use transformation \( T2 \) backwards. Denote by \( x = s - a, y = s - b, z = s - c \), where \( s \) is the semiperimeter of the triangle. The left-hand side is equivalent to
\[
\frac{x}{y + z} + \frac{y}{x + z} + \frac{z}{x + y} \geq \frac{3}{2},
\]
which a famous Nesbitt’s inequality, and we are done. ■

8. Let \( a, b, c \in (0, 1) \) be real numbers such that \( ab + bc + ca = 1 \). Prove that
\[
\frac{a}{1 - a^2} + \frac{b}{1 - b^2} + \frac{c}{1 - c^2} \geq \frac{3}{4} \left( \frac{1 - a^2}{a} + \frac{1 - b^2}{b} + \frac{1 - c^2}{c} \right).
\]
(Calin Popa)

Solution. We apply substitution \( S1 \) \( cyc(a \equiv \tan \frac{A}{2}) \), where \( A, B, C \) are angles of a triangle. Because \( a, b, c \in (0, 1) \), it follows that \( \tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2} \in (0, 1) \), that is \( A, B, C \) are angles of an acute triangle. Note that
\[
cyc \left( \frac{a}{1 - a^2} = \frac{\sin \frac{4}{2}}{\cos \frac{4}{2}} = \tan A \right).
\]

Thus the inequality is equivalent to
\[
\tan A + \tan B + \tan C \geq 3 \left( \frac{1}{\tan A} + \frac{1}{\tan B} + \frac{1}{\tan C} \right).
\]

Now observe that if we apply transformation \( T1 \) and the result in Theorem 1, we get
\[
\tan A + \tan B + \tan C = \tan A \tan B \tan C.
\]

Hence our inequality is equivalent to
\[
(\tan A + \tan B + \tan C)^2 \geq 3 (\tan A \tan B + \tan B \tan C + \tan A \tan C).
\]

This can be written as
\[
\frac{1}{2} (\tan A - \tan B)^2 + (\tan B - \tan C)^2 + (\tan C - \tan A)^2 \geq 0,
\]
and we are done. ■
9. Let $x, y, z$ be positive real numbers. Prove that

$$
\sqrt{\frac{y + z}{x}} + \sqrt{\frac{z + x}{y}} + \sqrt{\frac{x + y}{z}} \geq \sqrt{\frac{16(x + y + z)^3}{3(x + y)(y + z)(z + x)}}.
$$

*(Vo Quoc Ba Can, Mathematical Reflections, 2007)*

**Solution.** Note that the inequality is equivalent to

$$
\sum_{\text{cyc}} (y + z) \sqrt{\frac{(x + y)(z + x)}{x(x + y + z)}} \geq \frac{4(x + y + z)}{\sqrt{3}}.
$$

Let use transformation $T2$ and substitution $S8$. We get

$$
cyc \left( (y + z) \sqrt{\frac{(x + y)(z + x)}{x(x + y + z)}} \right) = \frac{a}{\cos \frac{a}{2}} = 4R \sin \frac{\alpha}{2},
$$

and

$$
\frac{4(x + y + z)}{\sqrt{3}} = \frac{4R (\sin \alpha + \sin \beta + \sin \gamma)}{\sqrt{3}},
$$

where $\alpha, \beta, \gamma$ are angles of a triangle with circumradius $R$. Therefore it suffices to prove that

$$
\frac{\sqrt{3}}{2} \left( \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \right) \geq \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + \sin \frac{\beta}{2} \cos \frac{\beta}{2} + \sin \frac{\gamma}{2} \cos \frac{\gamma}{2}.
$$

Because $f(x) = \cos \frac{x}{2}$ is a concave function on $[0, \pi]$, from Jensen’s inequality we obtain

$$
\frac{\sqrt{3}}{2} \geq \frac{1}{3} \left( \cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \right).
$$

Finally, we observe that $f(x) = \sin \frac{x}{2}$ is an increasing function on $[0, \pi]$, while $g(x) = \cos \frac{x}{2}$ is a decreasing function on $[0, \pi]$. Using Chebyshev’s inequality, we have

$$
\frac{1}{3} \left( \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \right) \left( \cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \right) \geq \sum \sin \frac{\alpha}{2} \cos \frac{\alpha}{2},
$$

and the conclusion follows. ■
Problems for independent study

1. Let $a, b, c$ be positive real numbers such that $a + b + c = 1$. Prove that
   \[
   \frac{a}{\sqrt{b + c}} + \frac{b}{\sqrt{c + a}} + \frac{c}{\sqrt{a + b}} \geq \frac{\sqrt{3}}{2}.
   \]
   (Romanian Mathematical Olympiad, 2005)

2. Let $a, b, c$ be positive real numbers such that $a + b + c = 1$. Prove that
   \[
   \sqrt{\frac{1}{a} - 1} \sqrt{\frac{1}{b} - 1} + \sqrt{\frac{1}{b} - 1} \sqrt{\frac{1}{c} - 1} + \sqrt{\frac{1}{c} - 1} \sqrt{\frac{1}{a} - 1} \geq 6.
   \]
   (A. Teplinsky, Ukraine, 2005)

3. Let $a, b, c$ be positive real numbers such that $ab + bc + ca = 1$. Prove that
   \[
   \frac{1}{\sqrt{a + b}} + \frac{1}{\sqrt{b + c}} + \frac{1}{\sqrt{c + a}} \geq 2 + \frac{1}{\sqrt{2}}.
   \]
   (Le Trung Kien)

4. Prove that for all positive real numbers $a, b, c$,
   \[
   (a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca).
   \]
   (APMO, 2004)

5. Let $x, y, z$ be positive real numbers such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Prove that
   \[
   \sqrt{x + yz} + \sqrt{x + yz} + \sqrt{x + yz} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.
   \]
   (APMO, 2002)

6. Let $a, b, c$ be positive real numbers. Prove that
   \[
   \frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} \geq 4 \left( \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \right).
   \]
   (Mircea Lascu)

7. Let $a, b, c$ be positive real numbers, such that $a + b + c = \sqrt{abc}$. Prove that
   \[
   ab + bc + ca \geq 9(a + b + c).
   \]
   (Belarus, 1996)
8. Let $a, b, c$ be positive real numbers. Prove that

$$\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} + 2\sqrt{\frac{abc}{(a + b)(b + c)(c + a)}} \geq 2$$

(Bui Viet Anh)

9. Let $a, b, c$ be positive real numbers such that $a + b + c = abc$. Prove that

$$(a - 1)(b - 1)(c - 1) \leq 6\sqrt{3} - 10.$$  

(Gabriel Dospinescu, Marian Tetiva)

10. Let $a, b, c$ be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$0 \leq ab + bc + ca - abc \leq 2.$$  

(Titu Andreescu, USAMO, 2001)

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